Large Deflection of Composite Beams

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Abstract

In this study, finite element method is used to compute deflections of composite beams undergoing geometric nonlinearity. The beams analyzed are laminates with different lay – ups and different end conditions, subjected to uniformly distributed loads. Integrals encountered in the analysis are performed by hand, and therefore considered more accurate than numerical integration. The results obtained showed excellent agreement with those found in literature. Extra results are generated to serve as bench marks for further investigations. As expected, large deflections have resulted in stiffer beams. The stiffness increase is more pronounced in beams which are less restrained.

Keywords: *Finite element method, large deflection, laminated composite beams.*

1. INTRODUCTION

Composite materials, or composites, are materials consisting of two or more phases. They have been used in engineering structures over many decades. Applications of composites are found in civil and mechanical engineering structures which include aircraft wings, wind turbines, robot arms, etc. Their advantages over metals include high strength/ weight ratio, high stiffness, high fatigue, high corrosion resistance, and lower thermal expansion. In addition to that the properties of composites can be tailored to meet the design requirements through selection of phase constituents that serve the end goal.

Composites are made of fiber embedded in a binding material or matrix. The fiber is the load carrying phase whereas the matrix protects and supports the fiber, and assists in transmitting external load to the fiber. Hence bonding between the fiber and the matrix is of utmost importance. In case of debonding or separation between fiber and matrix, the load carrying capacity of a composite structure can lower drastically. In high performance composites, the fibers are long or continuous, whilst they are short in low or medium performance composite.

A simple composite structure is usually in the form of a laminate made up of a number of layers or plies stacked together in some preferable order or lay – up. The fiber in each ply may be oriented at some angle to the axis of the laminate to achieve the required overall properties.

The ever growing use of composites has led to tremendous advancements in the analysis of laminates under external loads. It is well known that shear deformation is more pronounced in composites than in metals due to their low shear modulus to in – plane elastic modulus. This led to the evolution of a number of mathematical models which include the classical theory that ignores shear deformation [1], first – order shear deformation theory [2] that assumes constant shear deformation across the laminate's thickness, and a variety of higher – order shear deformation theories [3] that claim better representation of the deformed cross – sections.

The solution of the mathematical models ranges from close – form solutions to numerical solutions. The most popular numerical technique is the finite element method [4]. In the vast majority of analyses, the laminate is replaced by a single layer [5] in which the displacement field through the entire thickness of the laminate is assumed. The equivalent laminate properties are then calculated and the behavior of the laminate is determined.

There is an extensive body of literature involving linear analysis of laminates such as beams plates. However, to some degree little that addresses the nonlinear response, and in particular beams. There are four types of nonlinearity. First: material nonlinearity which is caused by nonlinear stress – strain curve as seen in plastic rubber materials. Second: geometric nonlinearity which is caused by significant change between the initial and final geometry or position of an element of the material. Third: boundary condition nonlinearity which is caused by contact between different parts that produces disproportionate change in deformation. Fourth: loading nonlinearity which is caused by load changing over time. Geometric linearity, which is the subject of this paper, is usually referred to as large deflection. In fact, large deflections and rotations lead to a drastic change in the behavior of a structure due to stiffness and internal loads changes.

Geometric nonlinearity in laminates results from the nonlinear strain equations, where the transverse displacement is coupled to the axial strains. As a result, mid – plane stretching of the laminate may occur. Such stretching causes increase in the laminate stiffness. That is to say when the load is small and consequently the deflection, the laminate resists the load only with bending stiffness. After the load has caused some curvature, the deflected laminate exhibits membrane stiffness in addition to the original bending stiffness, and hence the laminate becomes much stiffer. Of course if the load is kept increasing, a stage will be reached when the structure loses its stiffness and fails to support the load.

Large deflection effects should be included in any structural analysis when deformations and rotations reach significant values. The displacements computed on the assumption of linear or small deflection theory, will be overestimated since the theory neglects membrane stresses that contribute to the stiffness of the structure.

It should be noted that nonlinear problems do not lend themselves easily to analytical solutions. The majority of works in this area employ some numerical technique, and in particular the finite element method. Nonlinear analysis of isotropic bodies can be found in [6] and [7]. Examples of nonlinear analysis of composites can be found in Refs. [8], [9], and [10].

2. MATHEMATICAL FORMULATION

When a laminate is experiencing large deformations due to loading it, the virtual strain energy can be stated as:

$$\delta U = \int_V \eta \varepsilon^T \sigma dV \qquad (1)$$

Whereε, σ, and V denote strain, stress and volume respectively. When finite element method is employed, the virtual strain is:

$$\delta\varepsilon = \bar{B}\delta a \qquad (2)$$

Where$\bar{B}$ is a function of the shape functions N_i and nodal displacements a. Hence equation (1) can be written as:

$$\delta U = \delta a^T \int_V \bar{B}\sigma dV \qquad (3)$$

The work done by forces f applied at the nodes,

$$\delta W = -\delta a^T f \qquad (4)$$

The total energy:

$$\delta U + \delta W = 0$$

$$i.e. \quad \int_V \bar{B}\sigma dV - f = 0 \quad (5)$$

We seek approximate solution to equation (5) such that the residual, R, is zero or small number.

$$i.e. \quad R = \int_V \bar{B}\sigma dV - f = 0 \quad (6)$$

We use the Newton – Raphson method so we minimize the residual to obtain the tangent stiffness matrix K_T.

$$K_T = \frac{dR}{da} = \int_V \frac{d}{da}(\bar{B}^T\sigma)dV = \int_V \frac{d\bar{B}^T}{da}\sigma dV + \int_V \bar{B}^T \frac{d\sigma}{da} dV$$

Note that

$$\frac{d\sigma}{da} = \frac{d\sigma}{d\varepsilon} \cdot \frac{d\varepsilon}{da}$$

but,

$$\frac{d\sigma}{d\varepsilon} = C\,, \qquad \text{and}\,\frac{d\varepsilon}{da} = \bar{B}$$

$$\therefore\ \frac{d\sigma}{da} = C\,\bar{B}$$

Where C is a matrix of the material properties.

$$\therefore K_T = \int_V \frac{d\bar{B}^T}{da}\sigma dV + \int_V \bar{B}\,C\,\bar{B}\,dV \quad (7)$$

or

$$K_T = K_\sigma + \bar{K}$$

where,

$$K_\sigma = \int_V \frac{d\bar{B}^T}{da}\sigma dV \quad \text{and}\,\bar{K} = \int_V \bar{B}^T\,C\,B\,dV$$

$\bar{B}$can be split into two components: Linear part (small deformation) B_s , and nonlinear part (large deformation) B_L .

$$\bar{B} = B_s + B_L(a) \qquad (8)$$

$B_s$is not a function of$a$

$$\therefore K_\sigma = \int_V \frac{dB_L^T}{da}\sigma dV \quad (9)$$

$$\bar{K} = \int_V (B_s + B_L)^T\,C(B_s + B_L)dV$$

$$\bar{K} = K_{ss} + K_{sL} + K_{Ls} + K_{LL} \quad (10)$$

where,

$$K_{ss} = \int_V B_s^T\,C\,B_s\,dV \qquad (11)$$

$$K_{sL} = \int_V B_s^T\,C\,B_L\,dV \qquad (12)$$

$$K_{Ls} = \int_V B_L^T\,C\,B_s\,dV \qquad (13)$$

$$K_{LL} = \int_V \bar{B}_L\,C\,B_L\,dV \qquad (14)$$

Hence the tangent stiffness matrix is

$$K_T = K_{ss} + K_{sL} + K_{Ls} + K_{LL} + K_\sigma \quad (15)$$

International Journal of Engineering Research And Advanced Technology (IJERAT)
ISSN:2454-6135 [Volume. 03 Issue.3, March– 2017]
www.sretechjournal.org

Now let us consider a beam of length L and cross – sectional areabh, made up of n layers perfectly bonded together. The beam is supported at the ends and subjected to a uniformly distributed load q (load/ unit length) as shown in figure 2.1 below.

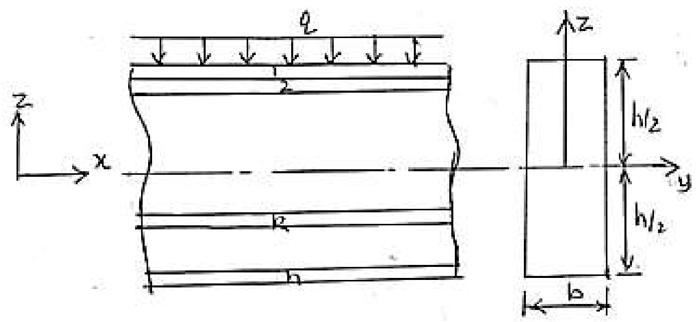

Figure 2.1

The beam is divided into m 2 – noded elements. Figure 2.2 shows a one typical element of length $H = L/m$, and the nodal displacements w_i and Q_i $(i = 1, 2)$.

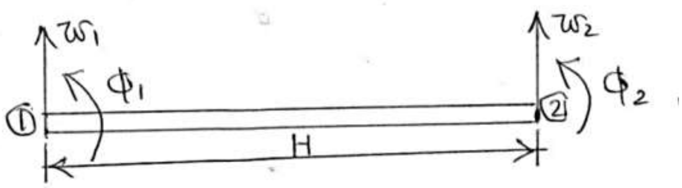

Figure 2.2

The deflection can be expressed in terms of the nodal displacements as:

$$w = N_1 w_1 + N_2 \phi_1 + N_3 w_2 + N_4 \phi_2 = N_i w_i \qquad (16)$$

where,

$$\phi = \frac{dw}{dx}$$

If we neglect the mid – plane deformation, we can write the longitudinal strain as:

$$\varepsilon = -z\frac{d^2 w}{dx^2} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \qquad (17)$$

The virtual strain:

$$\delta\varepsilon = -z\frac{d^2}{dx^2}\delta w + \left(\frac{dw}{dx}\right)\frac{d}{dx}\delta w = \bar{B}\delta w_i$$

where,

$$\bar{B} = \frac{d^2 N}{dx^2} + \left(\frac{dw}{dx}\right)\frac{dN}{dx} \qquad (18)$$

$$i.e.\ \ \bar{B} = B_s + B_L$$

where,

$$B_s = \frac{d^2N}{dx^2} \quad and \quad B_L = \left(\frac{dw}{dx}\right)\frac{dN}{dx}$$

Hence equations (11) – (14) become:

$$K_{ss} = bD\int_{L^e} \frac{d^2N_i}{dx^2}\frac{d^2N_j}{dx^2}\,dx \qquad (19)$$

$$K_{sL} = -\frac{bB}{2}\int_{L^e} \frac{d^2N_i}{dx^2}\left(\frac{dw}{dx}\right)\frac{dN_j}{dx}\,dx \qquad (20)$$

$$K_{Ls} = -\frac{bB}{2}\int_{L^e} \left(\frac{dw}{dx}\right)\frac{dN_i}{dx}\frac{d^2N_j}{dx^2}\,dx \qquad (21)$$

$$K_{Ls} = \frac{bA}{2}\int_{L^e} \left(\frac{dw}{dx}\right)^2\frac{dN_i}{dx}\frac{dN_j}{dx}\,dx \qquad (22)$$

where,

$$(A,B,D) = \sum_{k=1}^{n}\int_{Z_{k-1}}^{Z_k} C(1,Z,Z^2)\,dz \qquad (23)$$

Now to obtain K_σ , first rewrite equation (9) in terms of w_i

$$K_\sigma = \int_V \frac{dB_L^T}{dw_i}\sigma dV = \int_V \frac{d}{dw_i}\left(\frac{dw}{dx}\right)\frac{dN}{dx}\sigma dV$$

$$\therefore\ K_\sigma = \int_V \frac{dN_i}{dx}\frac{dN_j}{dx}\sigma dV \qquad (24)$$

let

$$\sigma = C\left[-Z\frac{d^2w}{dx^2} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] = C\left[-Z\frac{d^2N}{dx^2} + \frac{1}{2}\left(\frac{dw}{dx}\right)\frac{dN}{dx}\right] \qquad (25)$$

When equation (25) is substituted into equation (24), we get

$$K_\sigma = \frac{bA}{2}\int_{L^e}\left(\frac{dw}{dx}\right)^2\frac{dN_i}{dx}\frac{dN_j}{dx}\,dx - bB\int_{L^e}\left(\frac{d^2w}{dx^2}\right)\frac{dN_i}{dx}\frac{dN_j}{dx}\,dx \quad (26)$$

Up to now, the tangent stiffness matrix is known, so move on to the equation to be solved i.e. equation (5). When the load is uniformly distributed, and the rate of loading is q, f in equation (5) is replaced by $\int_{L^e} N^T dx$. Hence equation (5) reads as follows:

$$\int_{V^e} \bar{B}\sigma dV - \int_{L^e} N^T q\,dx = 0$$

Where $\sigma$from equation (25) and $\bar{B}$ from equation (18) are introduced in the equation above, and after integration is carried out, we get

$$K\,w_i - \int N^T q\,dx = 0 \qquad (27)$$

where,

$$K = K_{ss} + K_{sL} + K_{Ls} + K_{LL} \quad (28)$$

whereas

$$K_T = K_{ss} + K_{sL} + K_{Ls} + K_{\sigma} \qquad (29)$$

The solution procedure is as follows:

The solution is obtained using the Newton – Raphson iterative method as outlined in the following steps.

Step (1): set load f

Step (2): obtain the linear solution, w^o , as a first approximation.

Step (3): start iteration $k = 0 : 20$, say.

Step (4): establish matrix K_T from equation (29).

Step (5): compute correction as follows:

$$\delta w^k = -[K_T(w^k)]^{-1} R^k = -[K_T(w^k)]^{-1}[f - K(w^k)w^k]$$

Step (6): compute the new solution.

$$w^{k+1} = w^k + \delta w^k$$

Step (7): repeat iteration (steps 4 – 6) till the required accuracy is achieved. That is to say the error is less than a prescribed value according to the following law:

$$\sqrt{\frac{\sum_{i=1}^{n}\left(w_i^{k+1} - w_i^k\right)^2}{\sum_{i=1}^{n}\left(w_i^{k+1}\right)^2}} \le 10^{-3} \text{(in the present analysis)}$$

Local coordinates are used in the present study. Figure 2.3 shows an element in the local coordinate (r). The relation between the coordinate x and r is as follows:

$$x = x_o + \frac{H}{2} r \qquad \therefore dx = \frac{H}{2} dr$$

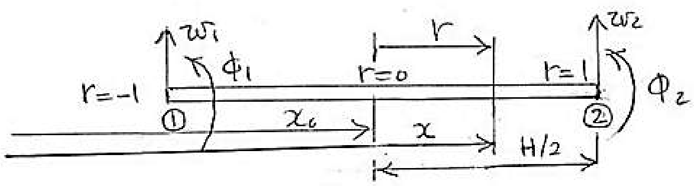

Figure 2.3

The shape function in local coordinates is expressed as follows:

$$N_i = a_i + b_i r + c_i r^2 + d_i r^3 \quad (i = 1, 2, 3, 4)$$

Where the coefficients are those given next:

i	a_i			
1	1/2	-3/4	0	1/4
2	1/4	-1/4	-1/4	1/4
3	1/2	3/4	0	-1/4
4	-1/4	-1/4	1/4	1/4

The deflection is expressed in terms of r

$$w = a + br + cr^2 + dr^3$$

where,

$$a = \frac{1}{2}(w_1 + w_2) + \frac{1}{4}(\phi_1 - \phi_2)$$

$$b = -\frac{3}{4}(w_1 - w_2) - \frac{1}{4}(\phi_1 + \phi_2)$$

$$c = -\frac{1}{4}(\phi_1 - \phi_2)$$

$$d = -\frac{1}{4}(w_1 - w_2) + \frac{1}{4}(\phi_1 + \phi_2)$$

The analysis is carried using non – dimensional quantities given next

$$\bar{w} = \frac{w}{h}\,, \qquad \bar{\phi} = \left(\frac{h}{L}\right)\phi\,, \qquad \bar{A} = \frac{A}{E_1 h}$$

$$\bar{B} = \frac{B}{E_1 h^2}\,, \qquad \bar{D} = \frac{D}{E_1 h^3}\,, \qquad \bar{q} = \left(\frac{L^5}{E_1 h^4}\right) q\,, \quad \bar{b} = \frac{b}{h}$$

The elements of the matrices given by equations (19) – (22) and equation (26) can be found in Appendix (A). As mentioned somewhere else.

$$(A, B, D) = \sum_{k=1}^{n} \int_{Z_{k-1}}^{Z_k} C(1, z, z^2) dz$$

$$C = C_{11}^1 C^4 + C_{22}^1 s^4 + 2(C_{12}^1 + 2C_{6\ 6}^1)S^2C^2$$

where,

$$C_{11}^1 = \frac{E_1}{1 - \nu_{12}\nu_{21}}\,, \qquad C_{12}^1 = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}\,, \qquad C_{6\ 6}^1 = G_{12}$$

and

$$s = sin\theta\,, \quad C = cos\theta$$

θ is the angle of orientation of the fiber in a certain ply; and n number of plies in a laminate:

E_1 and E_2 are the Young's Moduli of a lamina in the longitudinal and transverse directions respectively. G_{12} is the in – plane shear modulus.

3. RESULTS AND DISCUSSION

Amatlab computer program has been compiled for beams undergoing large deformations. All the results presented in this paper are for beams divided into ten elements. As explained earlier the Newton – Raphson method is employed in the analysis. It is worthwhile mentioning that convergence of solution was fast. The prescribed accuracy has been achieved after a few iterations.

To verify the accuracy of the present analysis, comparison is made with Ref. [11] where a built – in isotropic beam under uniform distributed load is considered. The beam is 2.5m in length, and 25 × 25mm in cross– section. The load varies from 0.2 to 2N/mm. The maximum non – dimensional deflections are given in Appendix (B): Table (1). It is evident that there is excellent agreement between the two sets of results. Therefore, we conclude that the present analysis is reasonably accurate in predicting the nonlinear behavior of beams in general.

Beams with four types of end conditions are considered, and these are: built – in beam (CC), simply supported beam (SS), cantilever beam (CF), and finally propped cantilever beam (CS). The maximum non – dimensional deflections of an isotropic beam under different loads are given in Appendix (B): Table (2). The effect of stiffening due to large deflection can easily be noticed. The percentage error or percentage over estimation of deflection according to linear analysis for a non – dimensional load of 40 is given in the next table (3.1) As a matter of fact, the percentage error represents percentage increase in the stiffness of the beam.

Table 3.1 Isotropic material; q=40

	CC	SS	CF	CS
Nonlinear	0.7773	1.0928	2.9908	0.9282
Linear	1.2480	5.7200	26.5000	2.5640
Error %	60.6	423.4	786.1	176.2

The increase in the stiffness of the cantilever beam (CF) is the most pronounced and stands at 786.1%, followed by the simply supported beam (SS) with 423.4% increase. The propped cantilever (CS) comes third with an increase of 176.2%. The least pronounced increase of stiffness is associated with the built – in beam (CC) which is 60.6%. However, it should be noted that very large increase in stiffness may not happen in real life. The beam that showed the largest increase in stiffness may not reach that level, and it may fail at a lower load due to stresses exceeding the safe limits.

The theoretical analysis as well as the computer program is set to solve beam problem irrespective of number of plies; orientation, or properties. However, only the results of symmetric cross – ply laminates are given. The laminates for which results have been generated are: $0, 0/90/0$, $0/90/90/0$, and $(0/90/90/0)_2$. The maximum non – dimensional deflections for these laminates for different end conditions and for a wide range of loads are given in Appendix (B):

Tables (3 – 5). Now assuming that the beams behave linearly when the load is unity, then we can estimate the deflections when q=40, and construct Tables (3.2 – 3.5) given next which show the difference between linear and nonlinear deflections for the different laminates and end conditions.

Table 3.2 Built – in beam (CC); q=40

	0	0/90/0	0/90/90/0	$(0/90/90/0)_2$
Nonlinear	0.7766	0.8559	0.9420	1.0749
Linear	1.2480	1.2960	1.4240	2.0960
Error %	60.7	51.4	51.2	95.0

Table 3.3 Simply supported beam (SS); q=40

	0	0/90/0	0/90/90/0	$(0/90/90/0)_2$
Nonlinear	1.0921	1.2386	1.3633	1.4100
Linear	5.7120	6.0480	6.6560	8.8600
Error %	423.0	388.3	388.2	528.4

Table 3.4 Cantilever beam (CF); q=40

	0	0/90/0	0/90/90/0	$(0/90/90/0)_2$
Nonlinear	2.9891	3.4073	3.7502	3.8082
Linear	26.4800	29.5360	32.5080	35.6280
Error %	785.9	766.8	766.8	835.6

Table 3.5Propped cantilever beam (CS); q=40

	0	0/90/0	0/90/90/0	$(0/90/90/0)_2$
Nonlinear	0.9275	1.0393	1.1438	1.2345
Linear	2.5600	2.6680	2.9360	4.2560
Error %	176.0	156.7	156.7	244.8

The percentage error in tables (3.2 – 3.5) represents the percentage increase of the stiffness. Laminates 0/90/0 and 0/90/90/0, with similar end conditions, differ in deflection, whereas they show equal percentage increase in stiffness. The effect of large deformation on the stiffness of a beam is more pronounced in the case of the eight ply laminate$(0/90/90/0)_2$. Figure 3.1 displays the deflection curves for a cantilever beam having three different lay – ups. The 90 degrees' plies seem to introduce softening effect to the laminate as one expects.

The support is one of the main factors that has significant effect on the deflection of a beam. The more the beam is restrained, the higher are the membrane stresses and the stiffer in the beam. However, the percentage increase in stiffness tends to rise as the beam becomes less restrained.

Figure 3.2 shows the deflection curves of the 8 – layers laminate when the beam is built – in and simply supported.

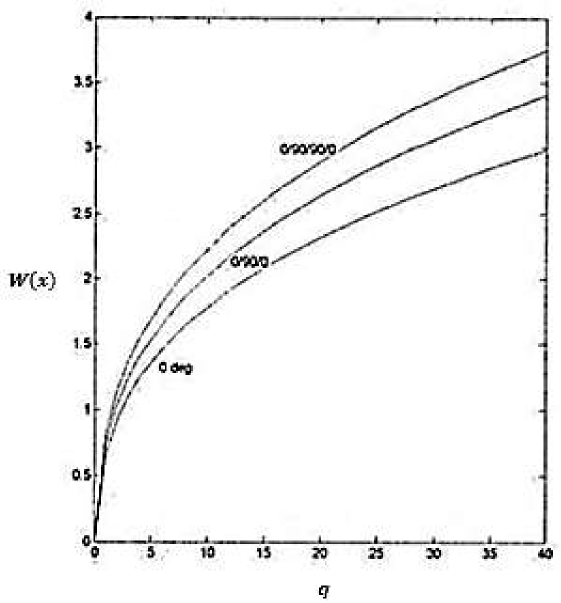

Figure 3.1 Center non – dimensional deflection for for a cantilever laminate

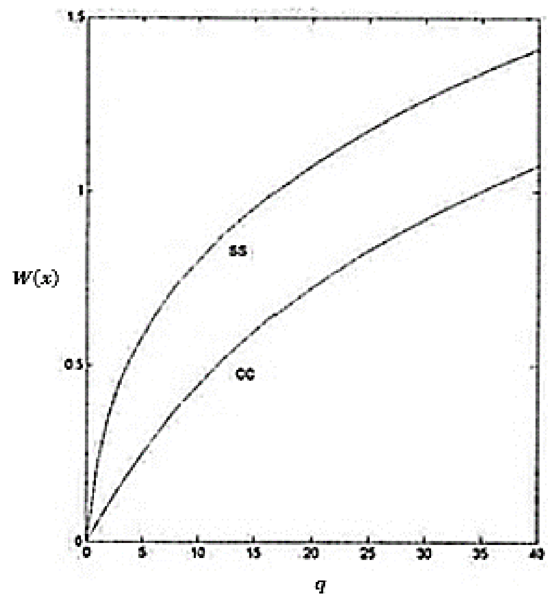

Figure 3.2 Center non – dimensional deflection for built – in and simply supported laminate

$(\mathbf{0/90/90/0})_2$

4. CONCLUSIONS

1. laminated beams subjected to large deflections have been studied. The nonlinear beam theory is presented, and a finite element computer program is compiled. Excellent agreement was found between the present work and similar ones in literature.
2. Large deflections lead to membrane stresses and consequently cause increase of the beam stiffness.
3. Neglecting large deflection effects can lead to over estimation of the deflection.
4. The end condition plays an important factor in the evaluation of deflection. The linear theory may overestimate the deflection of a built – in beam by one fold, but it may overestimate a less restrained beam such as a cantilever beam by eight folds.
5. The rate of change of stiffness depends on the lamination lay – up and the type of support. The lesser a beam is restrained; the greater is the stiffness change.

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APPENDICES

Appendix (A)

The elements of the different matrices are computed from the following expressions:

$$(K_{ss})_{i,j} = bD\int_{l^e} \frac{d^2N_i}{dx^2}\frac{d^2N_j}{dx^2}dx = bD \times \frac{8}{H^3}\int_{-1}^{1}\frac{d^2N_i}{dr^2}\frac{d^2N_j}{dr^2}dr = 8m^3bD\big(8C_iC_j + 24d_id_j\big)$$

$$(K_{sL})_{i,j} = -bB\int \frac{d^2N_i}{dx^2}\left(\frac{dw}{dx}\right)\left(\frac{dN_j}{dx}\right)dx = -bB \times \frac{8}{H^3}\int_{-1}^{1}\frac{d^2N_i}{dr^2}\left(\frac{dw}{dr}\right)\left(\frac{dN_j}{dr}\right)dr$$

$$= -8m^3bB\big[4b(C_iC_j + 2d_id_j + C_id_j) + 8C(d_ib_j + 2C_iC_j/3 + 9d_id_j/5) + 4d(C_ib_j + 18d_iC_j/5 + 9C_id_j/5)\big]$$

$$(K_{Ls})_{i,j} = -bB\int_{L^e}\left(\frac{dw}{dx}\right)\frac{dN_i}{dx}\frac{d^2N_i}{dx^2}dx = -bB \times \frac{8}{H^3}\int_{-1}^{1}\left(\frac{dw}{dr}\right)\frac{dN_i}{dr}\frac{d^2N_i}{dr^2}dr$$

$$= -8n^3bB\big[4b\big(b_iC_j + 2C_id_j + d_iC_j\big) + 8C(b_id_j + 2C_iC_j/3 + 9d_id_j/5) + 4d(b_iC + 18C_id_j/5 + 9d_iC_j/5)\big]$$

$$(K_{LL})_{i,j} = b\frac{A}{2}\int_{l^e}\left(\frac{dw}{dx}\right)^2\frac{dN_i}{dx}\frac{dN_j}{dx}dx = b\frac{A}{2}\times\frac{8}{H^3}\int_{-1}^{1}\left(\frac{dw}{dr}\right)^2\frac{dN_i}{dr}\frac{dN_j}{dr}dr$$

$$= 4m^3bA\big\{4b^2\big[b_ib_j + (3b_id_j + 2C_iC_j + 3d_ib_j)/3 + 9d_id_j/5\big] + 16bc\big[(b_iC_j + C_ib_j)/3 + 3(C_id_j + d_iC_j)/5\big]$$

$$+2(6bd + 4C^2)\big[b_ib_j/3 + (3b_id_j + 2C_iC_j + 3d_ib_j)/5 + 9d_id_j/7\big] + 48Cd\big[(b_iC_j + C_ib_j)/5 + 3(C_id_j + d_iC_j)/7\big]$$

$$+18d^2\left[b_ib_j/5+(3b_id_j+4C_iC_j+3d_ib_j)/7+d_id_j\right]$$

$$(K_\sigma)_{i,j}=b\frac{A}{2}\int_{L^e}\left(\frac{dw}{dx}\right)^2\frac{dN_i}{dx}\frac{dN_j}{dx}dx-bB\int_{L^e}\frac{d^2w}{dr^2}\frac{dN_i}{dx}\frac{dN_j}{dx}dx$$

$$=b\frac{A}{2}\times\frac{8}{H^3}\int_{-1}^{1}\left(\frac{dw}{dr}\right)^2\frac{dN_i}{dr}\frac{dN_j}{dr}dr-bB\times\frac{8}{H^3}\int_{-1}^{1}\frac{d^2w}{dr^2}\frac{dN_i}{dr}\frac{dN_j}{dr}dr$$

$$=4m^3bA\{4b^2\left[b_ib_j+(3b_id_j+2C_iC_j+3d_ib_j)/3+9d_id_j/5\right]+16bc\left[(b_iC_j+C_ib_j)/3+3(C_id_j+d_iC_j)/5\right]$$

$$+2(6bd+4C^2)\left[b_ib_j/3+(3b_id_j+4C_iC_j+3d_ib_j)/5+9d_id_j/7\right]+48Cd\left[(b_iC_j+C_ib_j)/5+3(C_id_j+d_iC_j)/7\right]$$

$$+18d^2\left[b_ib_j/5+(3b_id_j+4C_iC_j+3d_ib_j)/7+d_id_j\right]\}$$

$$-8m^3bB\{2C\left[2b_ib_j+2(3b_id_j+4C_iC_j+3d_ib_j)/3+18d_id_j/5\right]+6d\left[4(b_iC_j+C_ib_j)/3+12(C_id_j+d_iC_j)/5\right]\}$$

Appendix (B)

Table (1) Mid – span deflection (mm) of a built – in isotropic beam under a uniform distributed load $q(N/mm)$

q	Present	Ref. (11)	q	Present	Ref. (11)
0.2	3.0750	3.0752	1.2	14.1175	14.1140
0.4	5.9025	5.9046	1.4	15.6300	15.6227
0.6	8.3850	8.3892	1.6	17.0025	16.9947
0.8	10.5450	10.5524	1.8	18.2625	18.2549
1.0	12.4375	12.4358	2.0	19.4325	19.4220

Table (2) Maximum non – dimensional deflection of isotropic beams with different end conditions

q	CC	SS	CF	CS
1	0.0312	0.1430	0.6625	0.0641
8	0.2361	0.5470	1.6392	0.3837
16	0.4218	0.7495	2.1362	0.5835
24	0.5647	0.8893	2.4830	0.7214
32	0.6801	0.9998	2.7583	0.8334
40	0.7773	1.0928	2.9908	0.9282

Table (3) Non – dimensional mid – span deflection of built – in beam (CC) for different lamination lay – ups

q	0	0/90/0	0/90/90/0	(0/90/90/0)2
1	0.312	0.0324	0.0356	0.0524
8	0.2358	0.2481	0.2730	0.3742

16	0.4213	0.4517	0.4972	0.6272
24	0.5641	0.6129	0.6745	0.8047
32	0.6794	0.7445	0.8194	0.9540
40	0.7766	0.8559	0.9420	1.0749

Table (4) Non – dimensional mid – span deflection of a simply supported beam (SS) for different lamination lay – ups

q	0	0/90/0	0/90/90/0	(0/90/90/0)2
1	0.1428	0.1512	0.1664	0.2215
8	0.5466	0.6115	0.6730	0.7289
16	0.7490	0.8447	0.9297	0.9804
24	0.8887	1.0052	1.1063	1.1550
32	0.9992	1.1319	1.2458	1.2934
40	1.0921	1.2386	1.3633	1.4100

Table (5) Non – dimensional free end deflection of a cantilever beam (CF) for different lamination lay – ups

q	0	0/90/0	0/90/90/0	(0/90/90/0)2
1	0.6620	0.7384	0.8127	0.8907
8	1.6382	1.8592	2.0462	2.1110
16	2.1350	2.4287	2.6731	2.7342
24	2.4816	2.8259	3.1103	3.1699
32	2.7567	3.1411	3.4572	3.5163
40	2.9891	3.4073	3.7502	3.8082

Table (6) Maximum non – dimensional deflection of a propped cantilever beam (CS) for different lamination lay – ups

q	0	0/90/0	0/90/90/0	(0/90/90/0)2
1	0.0640	0.0667	0.0734	0.1064
8	0.3833	0.4169	0.4588	0.5488
16	0.5830	0.6457	0.7107	0.7974
24	0.7209	0.8044	0.8853	0.9751
32	0.8327	0.9304	1.0240	1.1160
40	0.9275	1.0393	1.1438	1.2345